EVA Tutorial #3

ISSUES ARISING IN EXTREME VALUE ANALYSIS

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Outline

(1) Penultimate Approximations

(2) Origin of Bounded and Heavy Tails

(3) Clustering at High Levels

(4) Complex Extreme Events

(5) Risk Communication under Stationarity

(6) Risk Communication under Nonstationarity

(1) Penultimate Approximations

- **"Ultimate" Extreme Value Theory**

-- GEV distribution as limiting distribution of maxima

$X_1, X_2, \ldots, X_n$ independent with common cdf F

$$M_n = \max\{ X_1, X_2, \ldots, X_n\}$$

- **Penultimate Extreme Value Theory**

-- Suppose F in domain of attraction of Gumbel type (i. e., $\xi = 0$)

-- Still preferable in nearly all cases to use GEV as approximate distribution for maxima (i. e., act as if $\xi \neq 0$)

-- Expression (as function of block size n) for shape parameter ξ_n

"Hazard rate" (or "failure rate"):

$$H_F(x) = F'(x) \,/\, [1 - F(x)]$$

Instantaneous rate of "failure" given "survived" until x

Alternative expression: $H_F(x) = -[\ln(1 - F)]'(x)$

One choice of shape parameter (block size n):

$$\xi_n = (1/H_F)'(x)\,\big|_{x=u(n)}$$

Here $u(n)$ is "characteristic largest value"

$$u(n) = F^{-1}(1 - 1/n)$$

[or $(1 - 1/n)$th quantile of F]

-- Because F assumed in domain of attraction of Gumbel,

$$\xi_n \longrightarrow 0 \text{ as block size } n \longrightarrow \infty$$

-- More generally, can use behavior of $H_F(x)$ for large x to determine domain of attraction of F

In particular, if

$$(1/H_F)'(x) \longrightarrow 0 \text{ as } x \longrightarrow \infty$$

then F is in domain of attraction of Gumbel

Note: Straightforward to show that hazard rate of lognormal distribution satisfies above condition (i. e., in domain of attraction of Gumbel)

- ***Example*: Exponential Distribution**

-- Exact exponential upper tail (unit scale parameter)

$$1 - F(x) = \exp(-x), \quad x > 0$$

-- Penultimate approximation

Hazard rate: $H_F(x) = 1, \quad x > 0$

(Constant hazard rate consistent with memoryless property)

Shape parameter: $\xi_n = 0$

So *no* benefit to penultimate approximation

- ***Example***: **Normal Distribution (with zero mean & unit variance)**

-- Fisher & Tippett (1928) proposed Weibull type of GEV as penultimate approximation

Hazard rate: $H_{\Phi}(x) \approx x$, **for large** x

[Recall that $1 - \Phi(x) \approx \varphi(x) / x$**]**

Characteristic largest value: $u(n) \approx (2 \ln n)^{1/2}$, **for large** n

Penultimate approximation is Weibull type with

$$\xi_n \approx -1 / (2 \ln n)$$

For example: $\xi_{100} \approx -0.11$, $\xi_{365} \approx -0.085$

- ***Example*: "Stretched Exponential" Distribution**

-- Traditional form of Weibull distribution (Bounded below)

$$1 - F(x) = \exp(-x^{c}),\ \ x > 0,\ \ c > 0$$

where c is shape parameter (unit scale parameter)

Hazard rate: $H_F(x) = c\, x^{c-1},\ \ x > 0$

Characteristic largest value: $u(n) = (\ln n)^{1/c}$

Penultimate approximation has shape parameter

$$\xi_n \approx (1 - c) / (c \ln n)$$

(i) $c > 1$ implies $\xi_n \uparrow 0$ as $n \to \infty$ (i. e., Weibull type)

(ii) $c < 1$ implies $\xi_n \downarrow 0$ as $n \to \infty$ (i. e., Fréchet type)

(2) Origin of Bounded and Heavy Tails

- **Upper Bounds / Penultimate approximation**

-- Weibull type of GEV (i. e., $\xi < 0$)

For instance, provides better approximation than Gumbel type when "parent" distribution *F*:

(i) Normal (e. g., for temperature)

(ii) Stretched exponential with *c* > 1 (e. g., for wind speed)

-- Apparent upper bound

Complicates interpretation (e. g., "thermostat hypothesis" or maximum intensity of hurricanes)

- **Heavy tails / Penultimate approximation**

-- Fréchet type of GEV (i. e., $\xi > 0$)

For instance, provides better approximation than Gumbel when parent distribution *F*:

Stretched exponential distribution with $c < 1$

-- Possible explanation for apparent heavy tail of precipitation

Wilson & Toumi (2005):

Based on physical argument, proposed stretched exponential with $c = 2/3$ (Universal value, independent of season or location) as distribution for heavy precipitation

-- Simulation experiment

Generated observations from stretched exponential distribution with shape parameter $c = 2/3$

Determine maximum of sequence of length $n = 100$, M_{100}

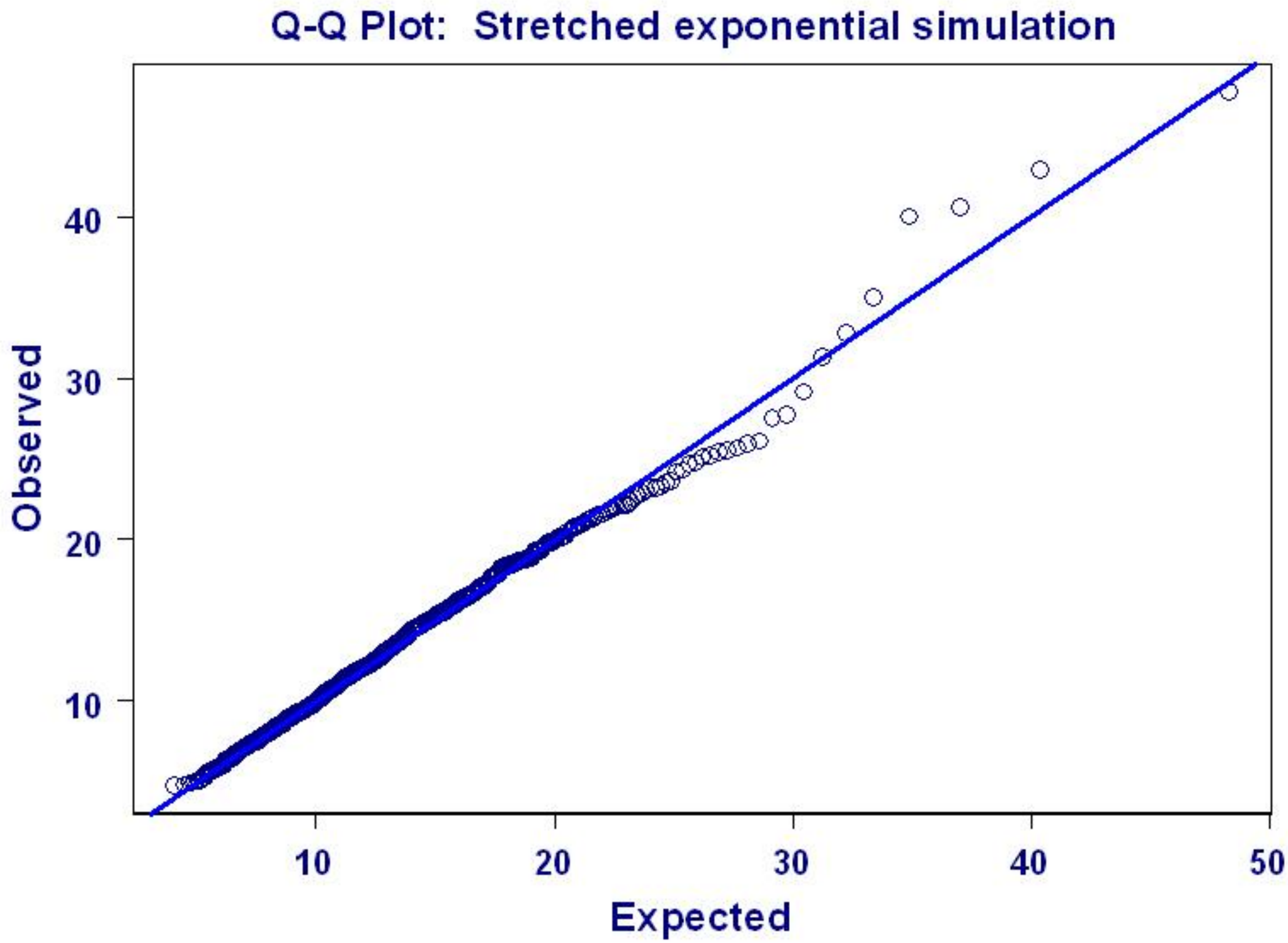
(Annual maxima: Daily precipitation occurrence rate ≈ 27%)

Annual prec. maxima: Typical estimated $\xi \approx 0.10$ to 0.15

(Penultimate approximation gives $\xi_{100} \approx 0.11$)

Fitted GEV distribution (Sample size = 1000):

Obtained estimate of $\xi \approx 0.12$


Q-Q Plot: Stretched exponential simulation
Observed
Expected
10
20
30
40
50

- **Heavy Tails / Chance mechanism**

-- Mixture of exponential distributions

Suppose *X* has exponential distribution with scale parameter σ*:

$$\Pr\{X > x \mid \sigma^*\} = \exp[-(x/\sigma^*)], \quad x > 0, \quad \sigma^* > 0$$

Further assume that the rate parameter $v = 1/\sigma^*$ varies according to a gamma distribution with shape parameter α (unit scale), pdf:

$$f_V(v; \alpha) = [\Gamma(\alpha)]^{-1} v^{\alpha-1} \exp(-v), \quad \alpha > 0$$

The unconditional distribution of *Y* is heavy-tailed:

$$\Pr\{X > x\} = (1 + x)^{-\alpha}$$

(i.e., exact GP distribution with shape parameter $\xi = 1/\alpha$)

-- Simulation experiment

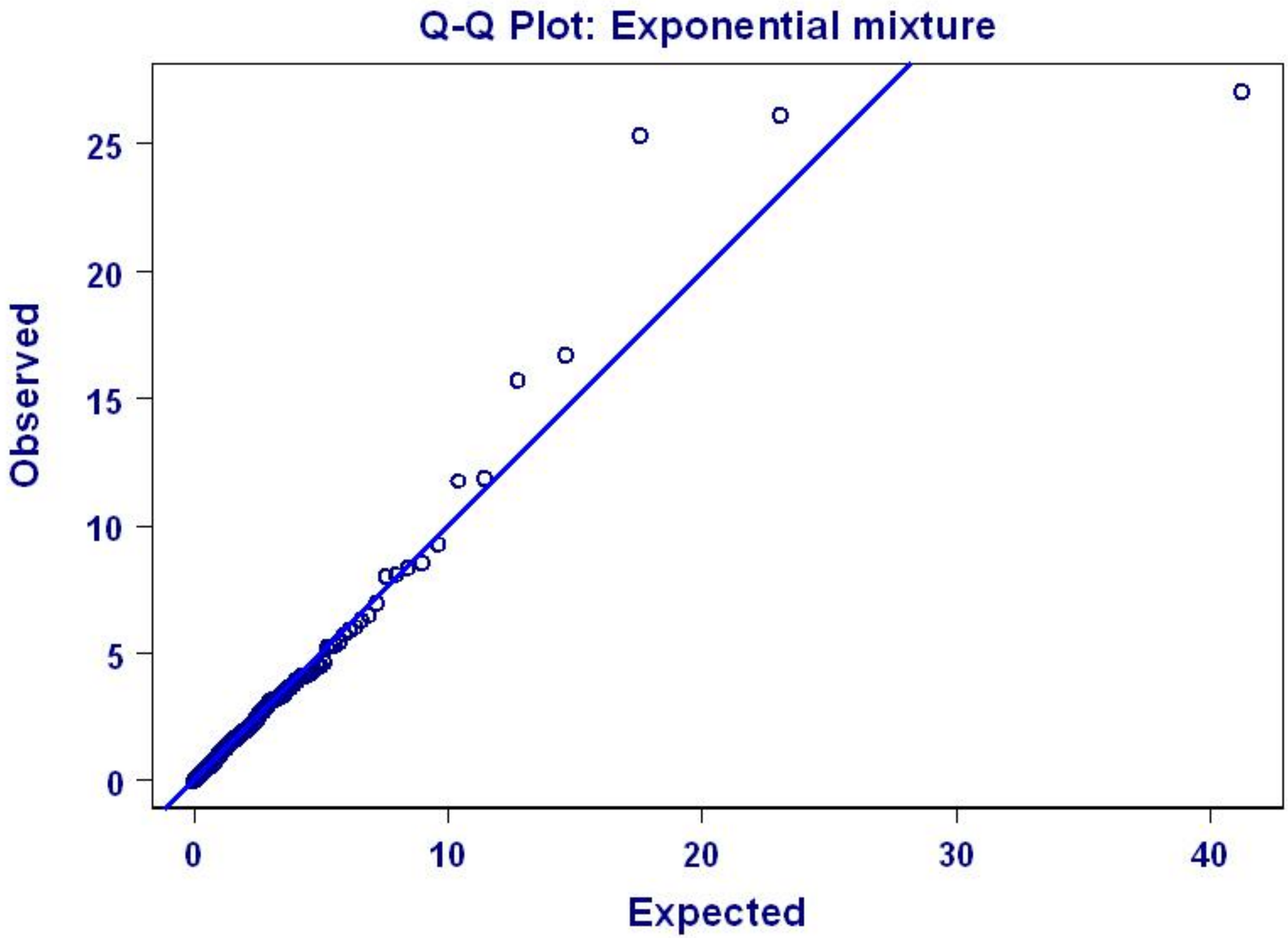
Induce heavy tail from conditional light tails

Let rate parameter of exponential distribution have gamma distribution with shape parameter $\alpha = 2$

Then unconditional (mixture) distribution is GP with shape parameter $\xi = 0.5$

Fit GP distribution to simulated exponential mixture (Sample size = 1000):

Obtained estimate of $\xi \approx 0.51$



Q-Q Plot: Exponential mixture
Observed
Expected
0
5
10
15
20
25
0
10
20
30
40

(3) Clustering at High Levels

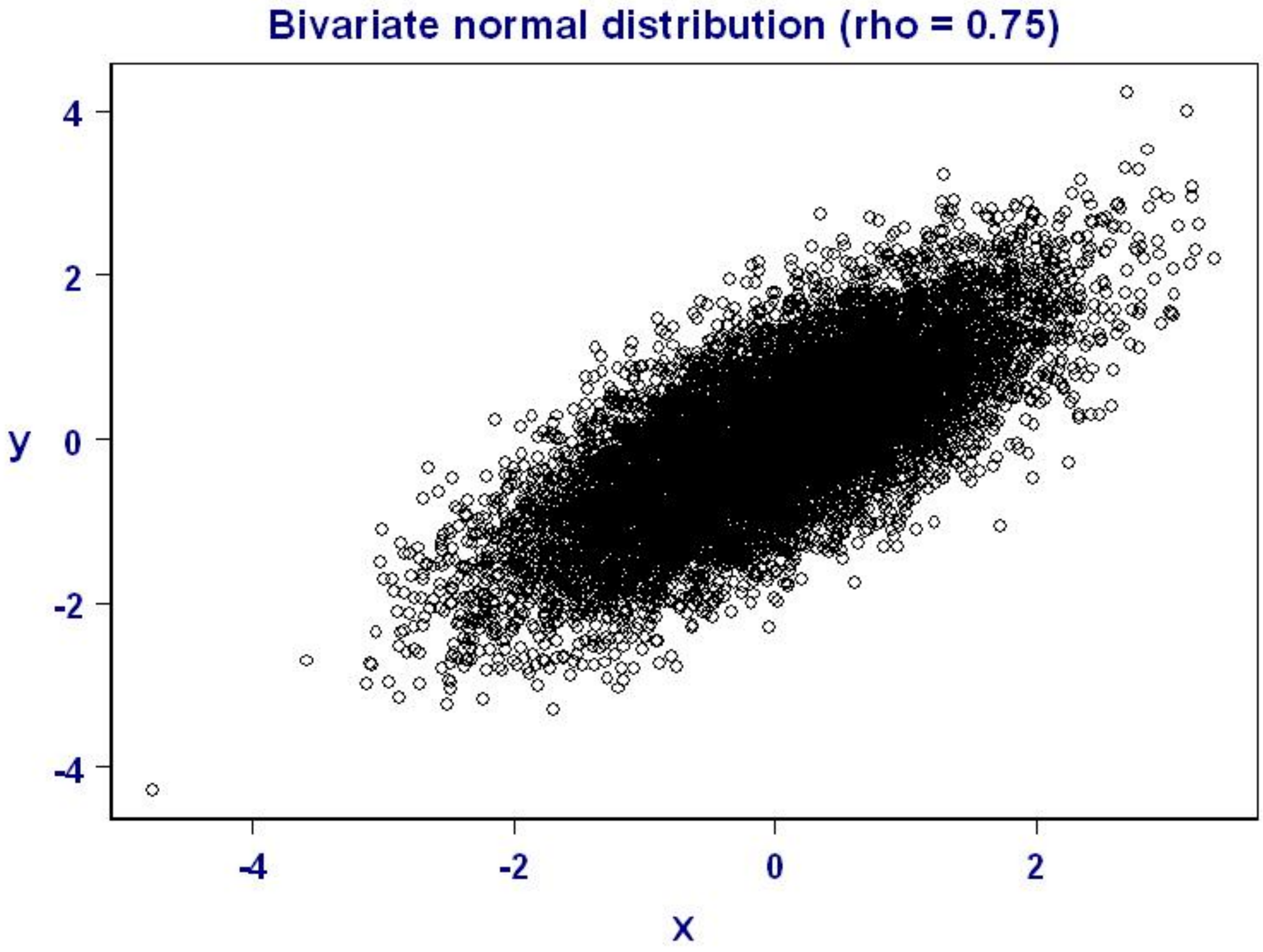
- **As example, consider stationary Gaussian process**

-- Joint distribution of X_t and X_{t+k} is bivariate normal with autocorrelation coefficient ρ_k, $k = 1, 2, \ldots$

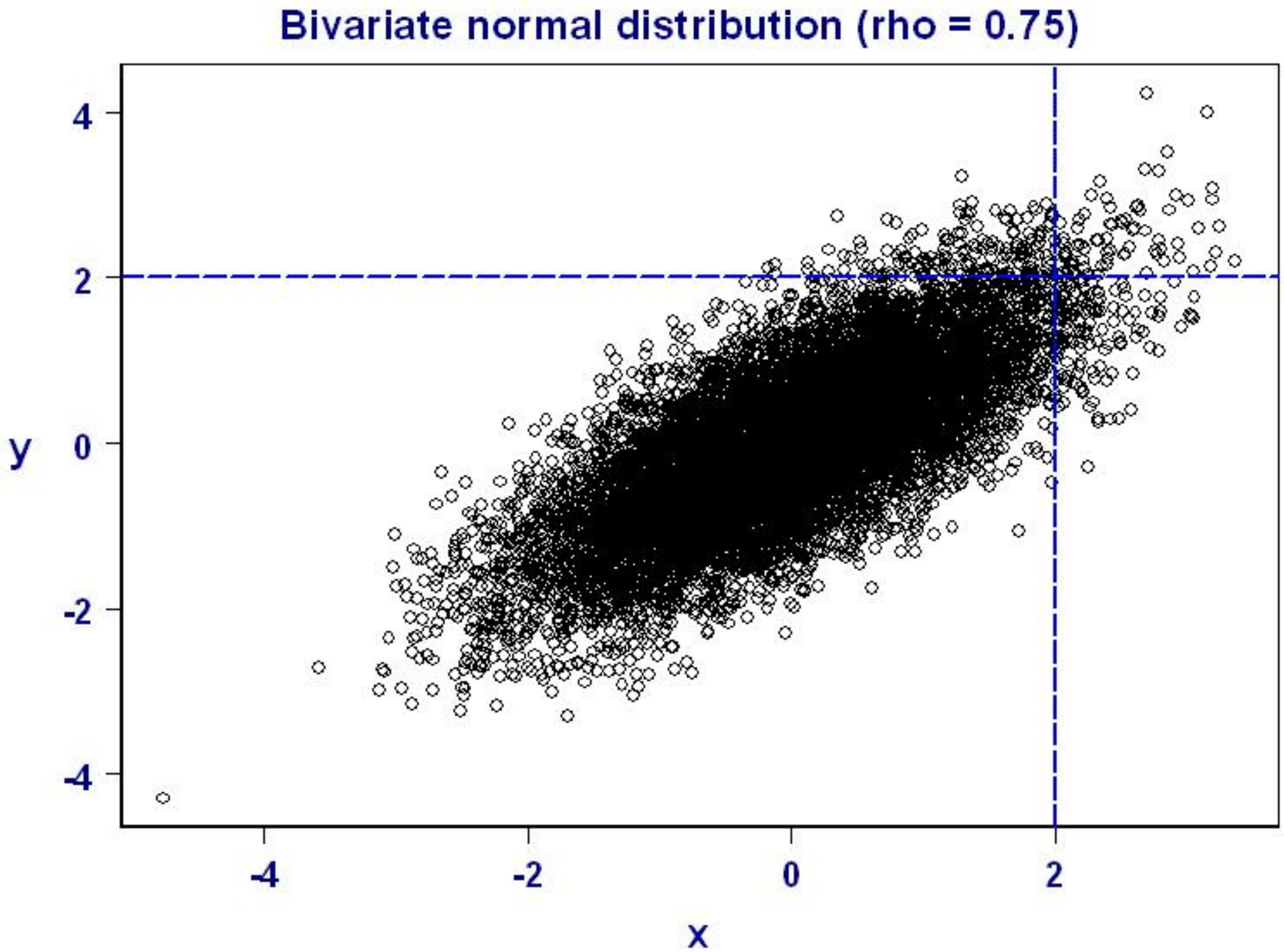
-- So consider two random variables (X, Y) with bivariate normal distribution with correlation coefficient ρ, $|\rho| < 1$

No "clustering at high levels" (in asymptotic sense; i. e., extremal index $\theta = 1$):

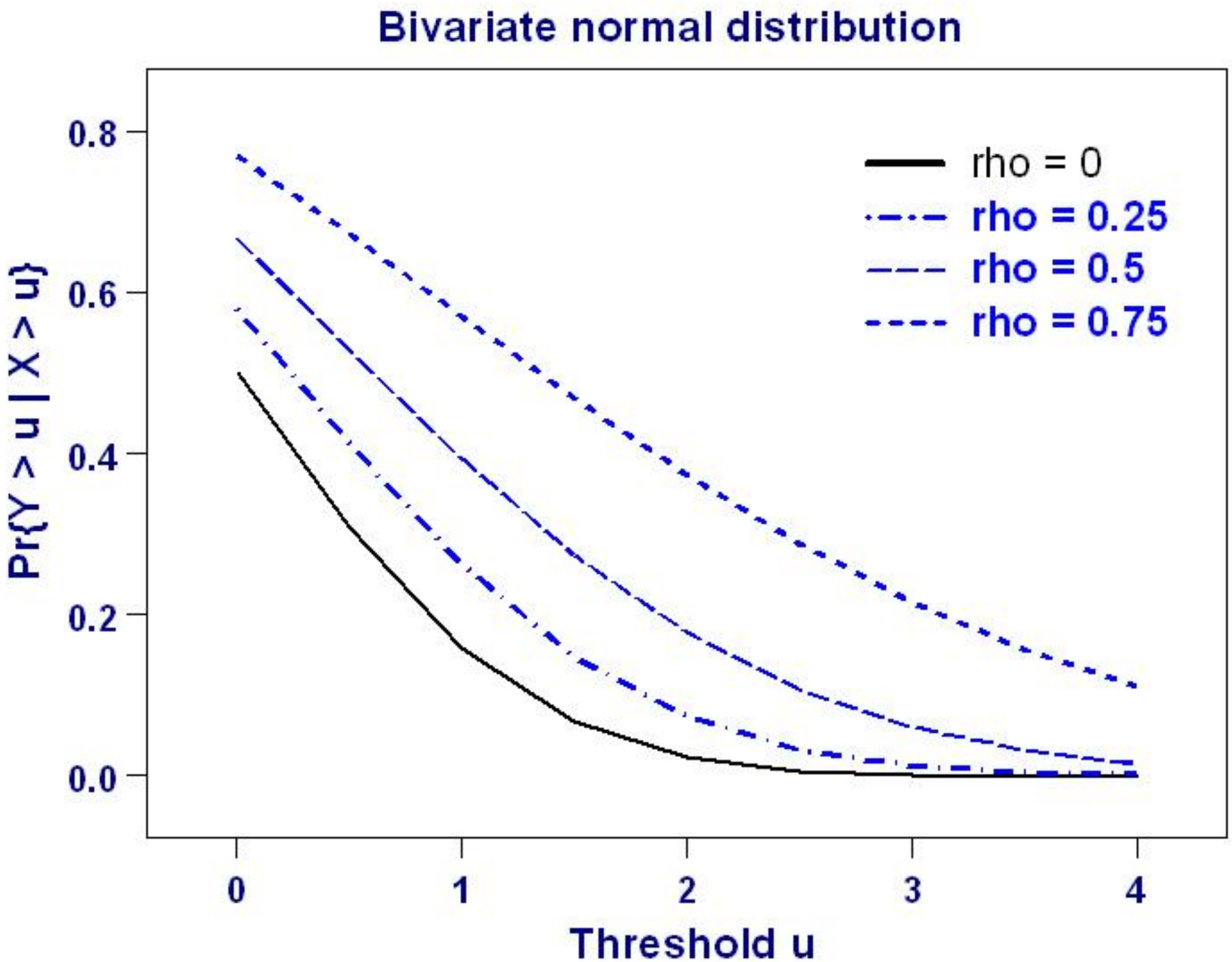
$$\Pr\{Y > u \mid X > u\} \to 0 \text{ as } u \to \infty$$



Simulation (sample size = 10,000)



Simulation (sample size = 10,000)


Bivariate normal distribution
rho = 0
rho = 0.25
rho = 0.5
rho = 0.75
Pr{Y > u | X > u}
0.0
0.2
0.4
0.6
0.8
0
1
2
3
4
Threshold u

- **Interpretation of extremal index θ, 0 < θ ≤ 1**

(i) Mean cluster length ≈ 1/θ

(ii) Effective sample size

(as if take maximum of *n = *n*θ "unclustered" observations)**

***Note*: Does not resemble same concept based on time averages**

Effect of θ < 1 on GEV distribution:

Adjustment to location and scale parameters, μ and σ, but no adjustment to shape parameter ξ

In block maxima approach, effect of θ < 1 automatically subsumed in fitted parameters of GEV (could affect approximation accuracy)

- **“Intervals estimator” of extremal index θ (Ferro-Segers 2003)**

-- “Interexceedance” times (i. e., time between exceedances)

(i) If $X_t > u$ & $X_{t+1} > u$, then interexceedance time = 1

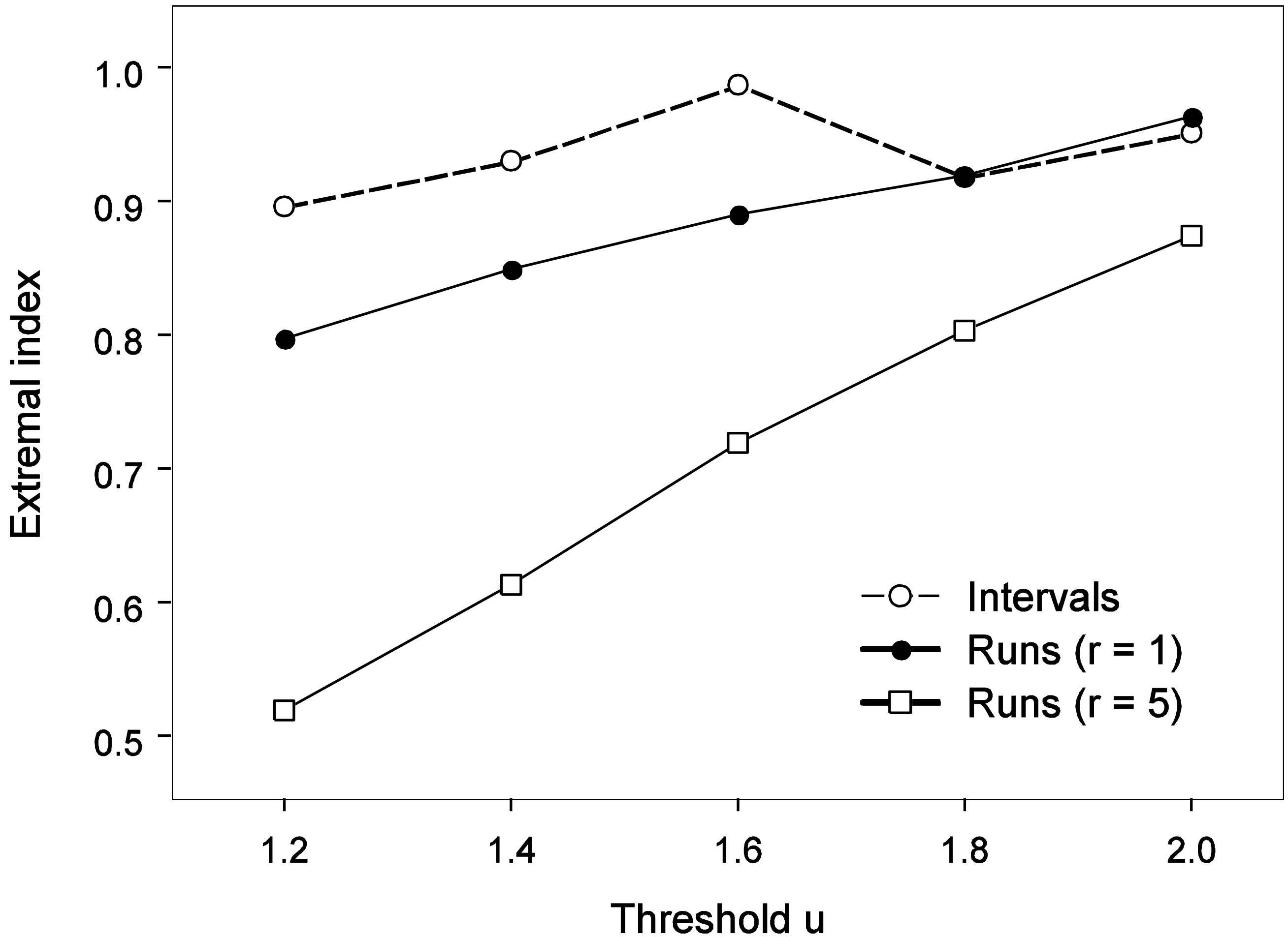
(ii) If $X_t > u$, $X_{t+1} < u$, $X_{t+2} > u$, then interexceedance time = 2, etc.

Coefficient of variation (i. e., st. dev. / mean) of interexceedance times converges to function of θ as threshold $u \to \infty$

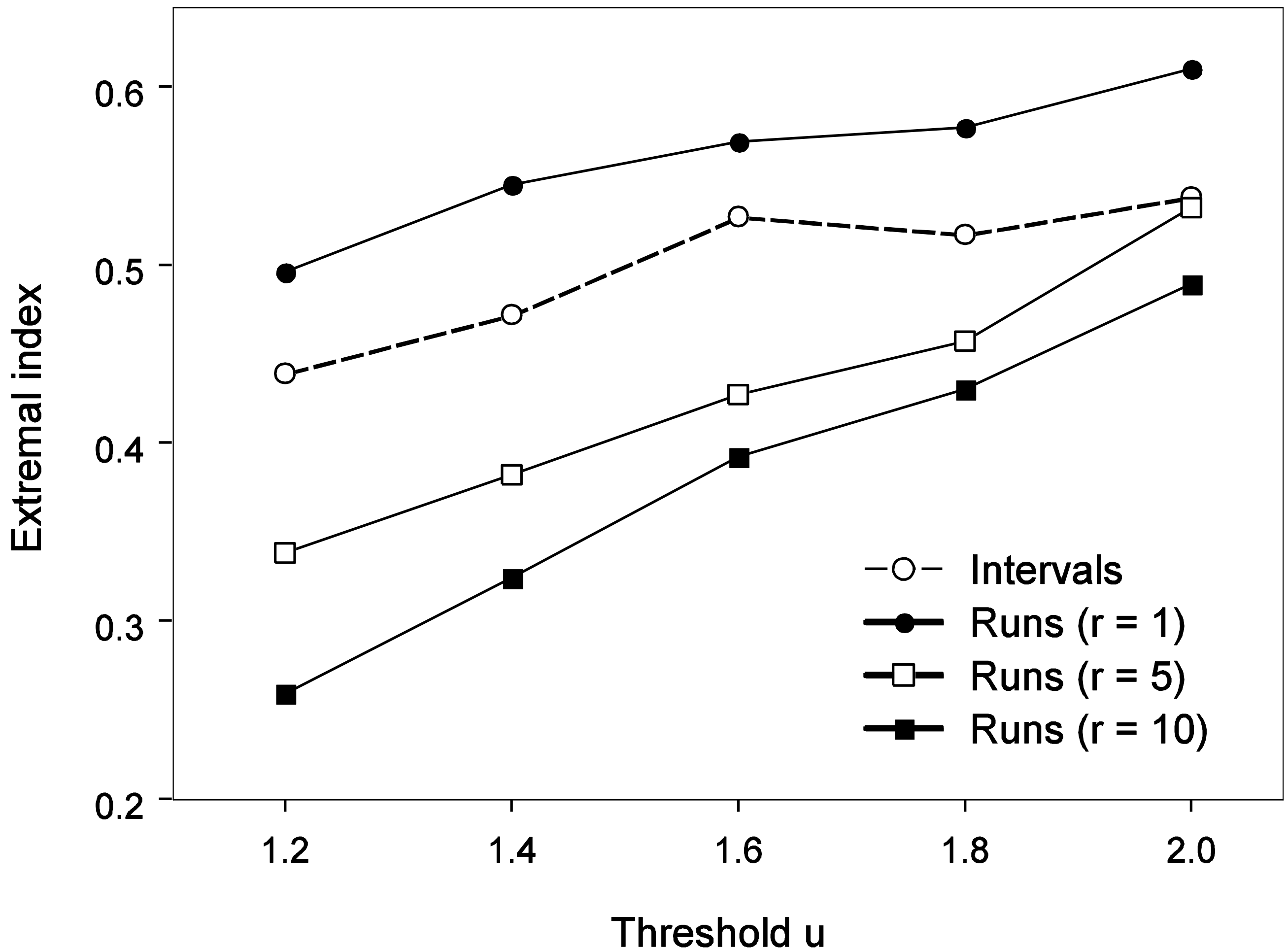
Does *not* require identification of clusters (could chose runs declustering parameter *r* so that mean cluster length ≈ 1/θ)

-- Confidence interval for θ

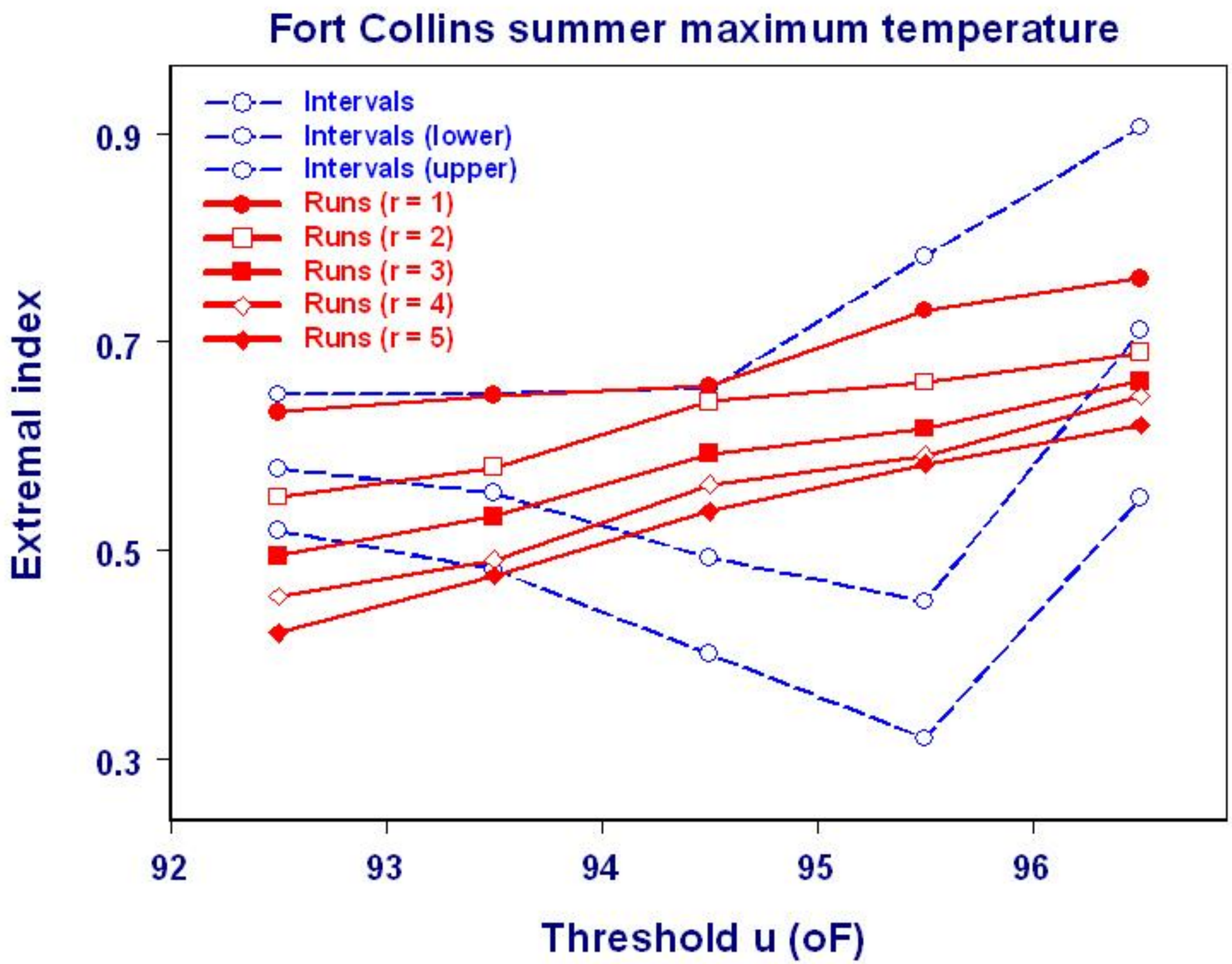
Resample interexceedance times (because of extremal dependence, need to modify conventional bootstrap)



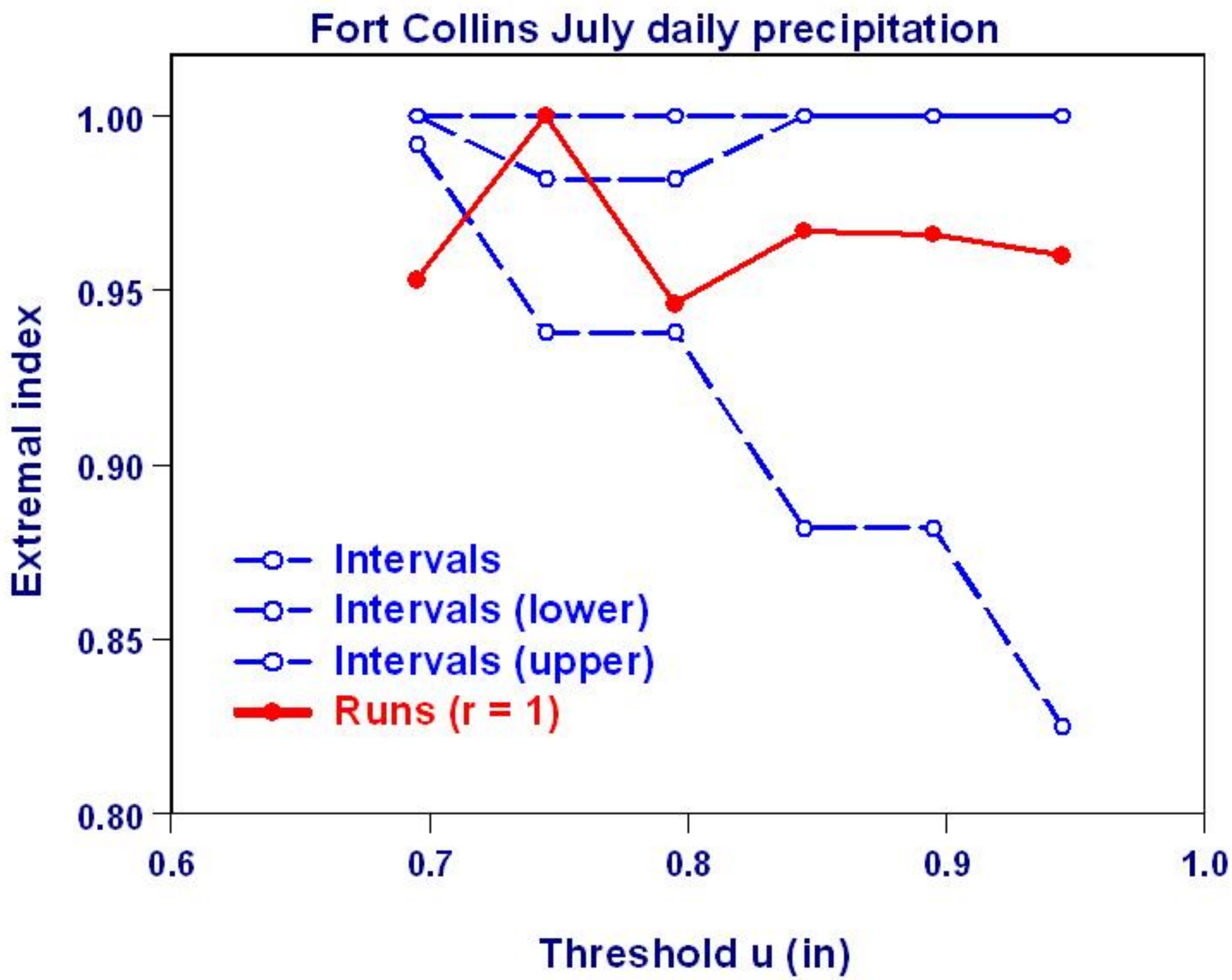
Gaussian first-order autoregressive process with $\rho_1 = 0.25$



Gaussian first-order autoregressive process with ρ_1 = 0.75



Evidence of clustering at high levels



Lack of evidence of clustering at high levels

(4) Complex Extreme Events

- **Heat waves**

-- Extreme weather phenomenon

-- Lack of use of statistical methods based on extreme value theory

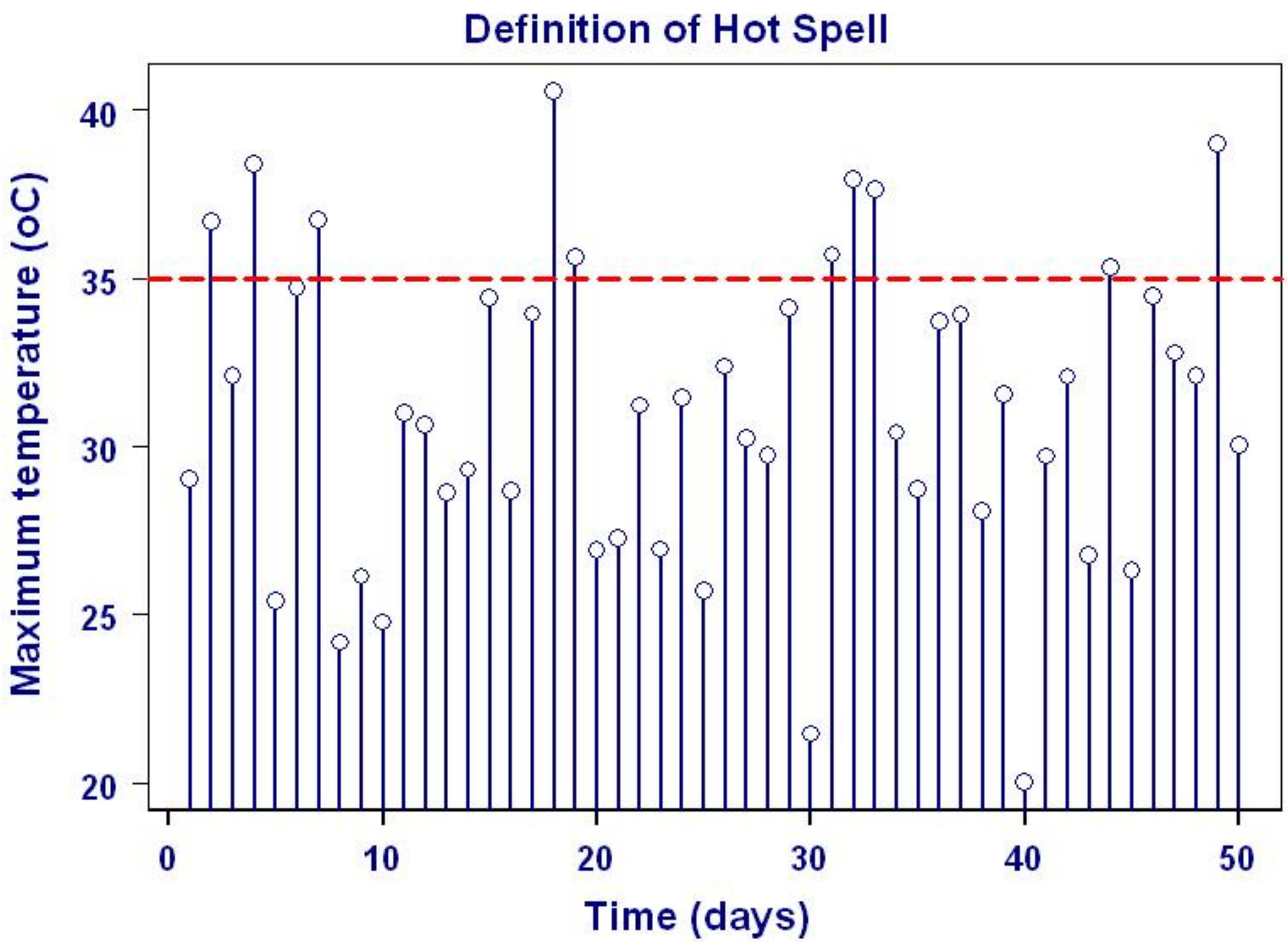
-- Complex phenomenon / Ambiguous concept

-- Focus on hot spells instead

(Derive more full-fledged heat waves from model for hot spells)

-- Devise simple model (only use univariate extreme value theory)

-- Simple enough to incorporate trends (or other covariates)


Definition of Hot Spell
Maximum temperature (oC)
Time (days)
20
25
30
35
40
0
10
20
30
40
50

- **Start with point process (or Poisson-GP) model**

-- Rate of occurrence of clusters

Modeled as Poisson process (rate parameter λ)

-- Intensity of cluster

Cluster maxima modeled as GP distribution (shape parameter ξ, scale parameter σ*)

- **Retain clusters ("hot spells"), rather than declustering**

-- Model cluster statistics

(i) Duration (e. g., geometric distribution with mean 1/θ)

(ii) Dependence of excesses within cluster (conditional GP model)

- **Model for excesses with cluster (runs parmeter r = 1)**

-- Let Y_1, Y_2, . . ., Y_k denote excesses over threshold within given cluster / spell (assume of length $k > 1$)

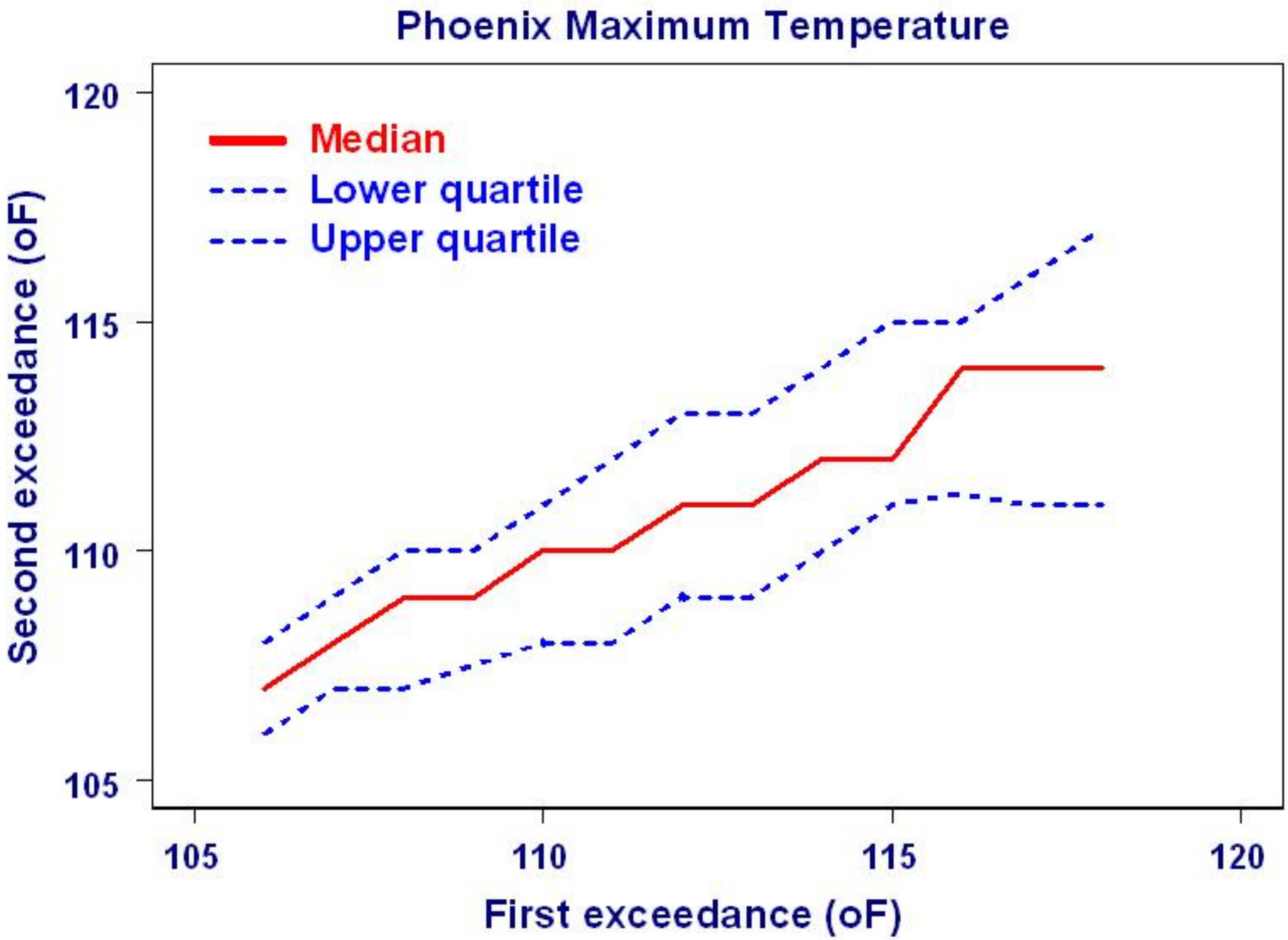
(i) Model first excessY_1 as unconditional GP distribution (instead of cluster maxima)

(ii) Model conditional distribution of Y_2 given Y_1 as GP with scale parameter depending on Y_1; e. g., with linear link function

$$\sigma^*(y) = \sigma_0^* + \sigma_1^* \, y, \quad \text{given } Y_1 = y$$

Similar model for conditional distribution of Y_3 given Y_2 (etc.)

Requires only univariate extreme value theory (not multivariate)


Phoenix Maximum Temperature
Median
Lower quartile
Upper quartile
120
115
110
105
Second exceedance (oF)
105
110
115
120
First exceedance (oF)

- **Conditional distribution of Y_2 given $Y_1 = y$**

-- Conditional mean [increases with σ*(*y*)]

$$E(Y_2 \mid Y_1 = y) = \sigma^*(y) / (1 - \xi), \quad \xi < 1$$

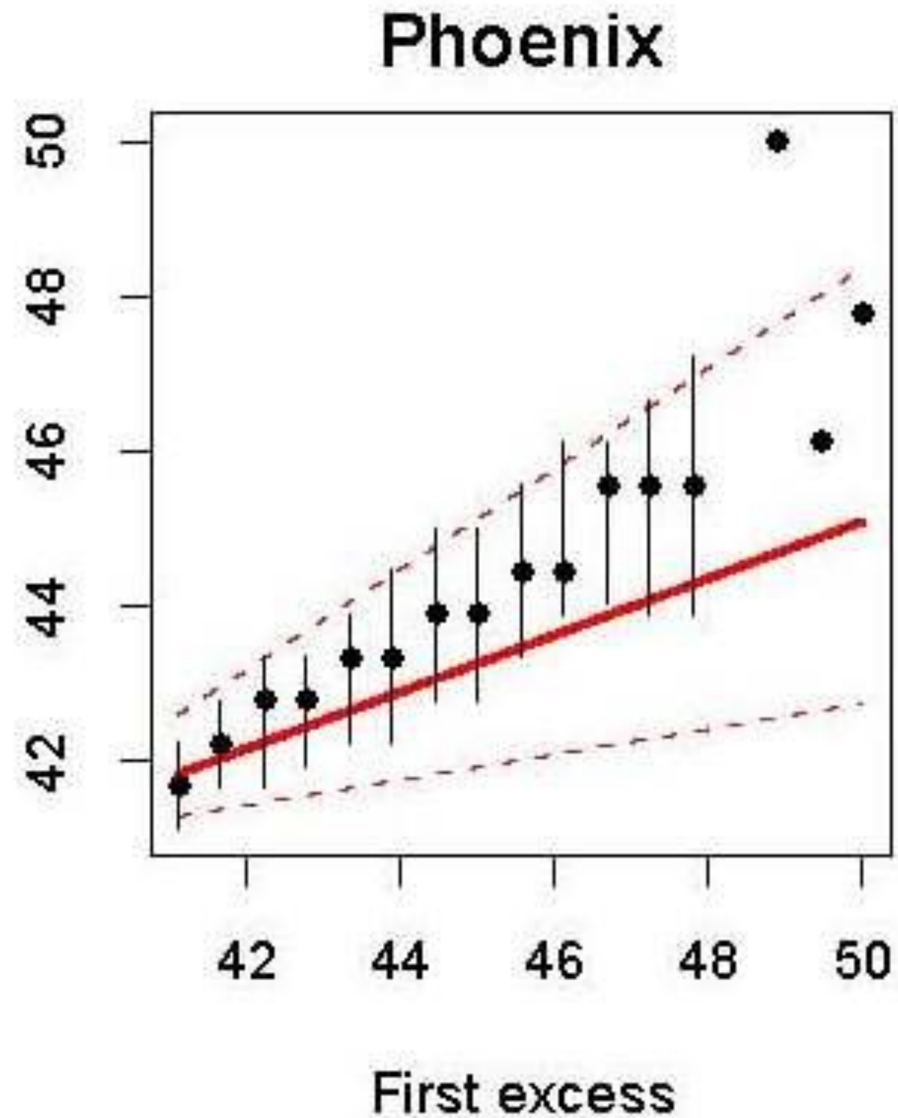
-- Conditional variance (increases with mean)

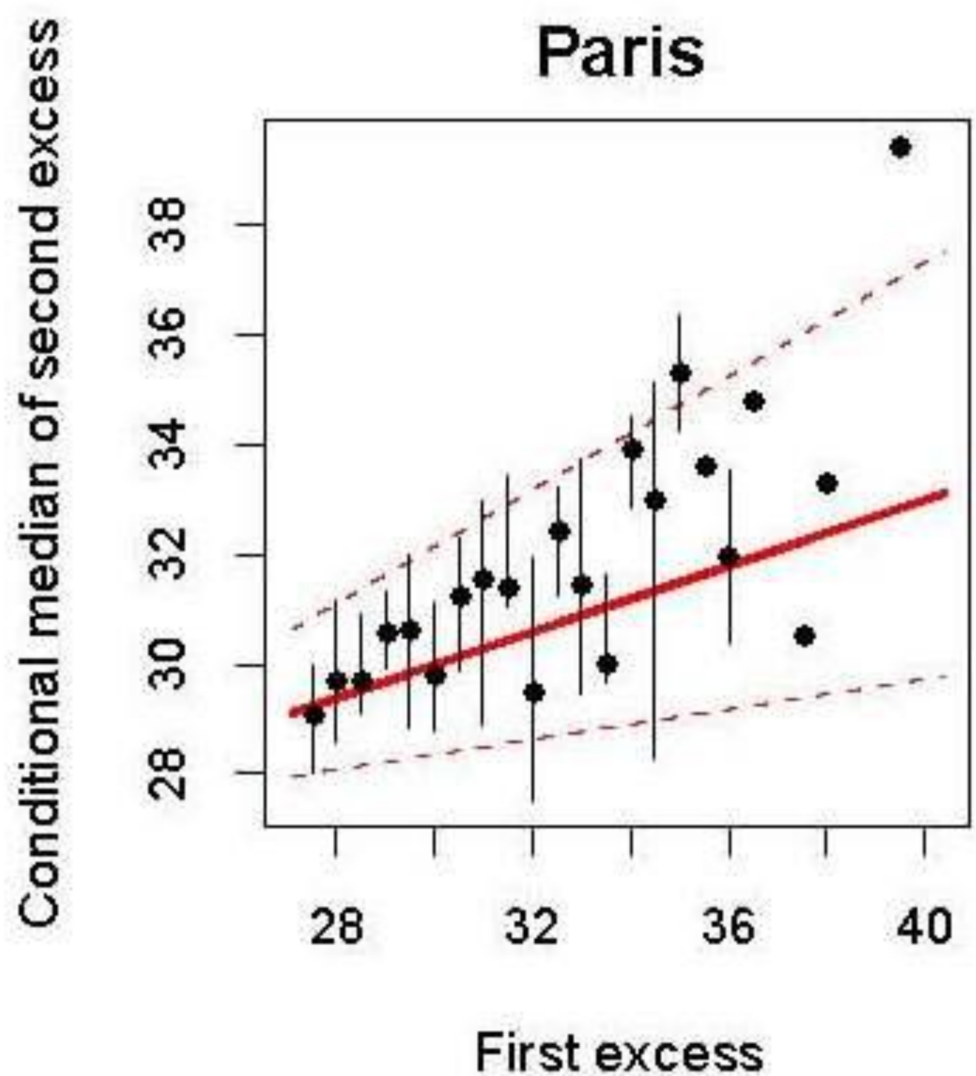
$$\mathrm{Var}(Y_2 \mid Y_1 = y) = [E(Y_2 \mid Y_1 = y)]^2 / (1 - 2\xi), \quad \xi < 1/2$$

-- Conditional quantile function

$$F^{-1}[p; \sigma^*(y), \xi] = [\sigma^*(y) / \xi] \, [(1 - p)^{-\xi} - 1], \quad 0 < p < 1$$

Increases more rapidly with σ*(*y*) for higher *p*


Phoenix
50
48
46
44
42
42
44
46
48
50
First excess


Paris
Conditional median of second excess
28
30
32
34
36
38
28
32
36
40
First excess

- **Introduction of trends**

-- Cluster rate

Trend in mean of Poisson rate parameter $\lambda(s)$, year s

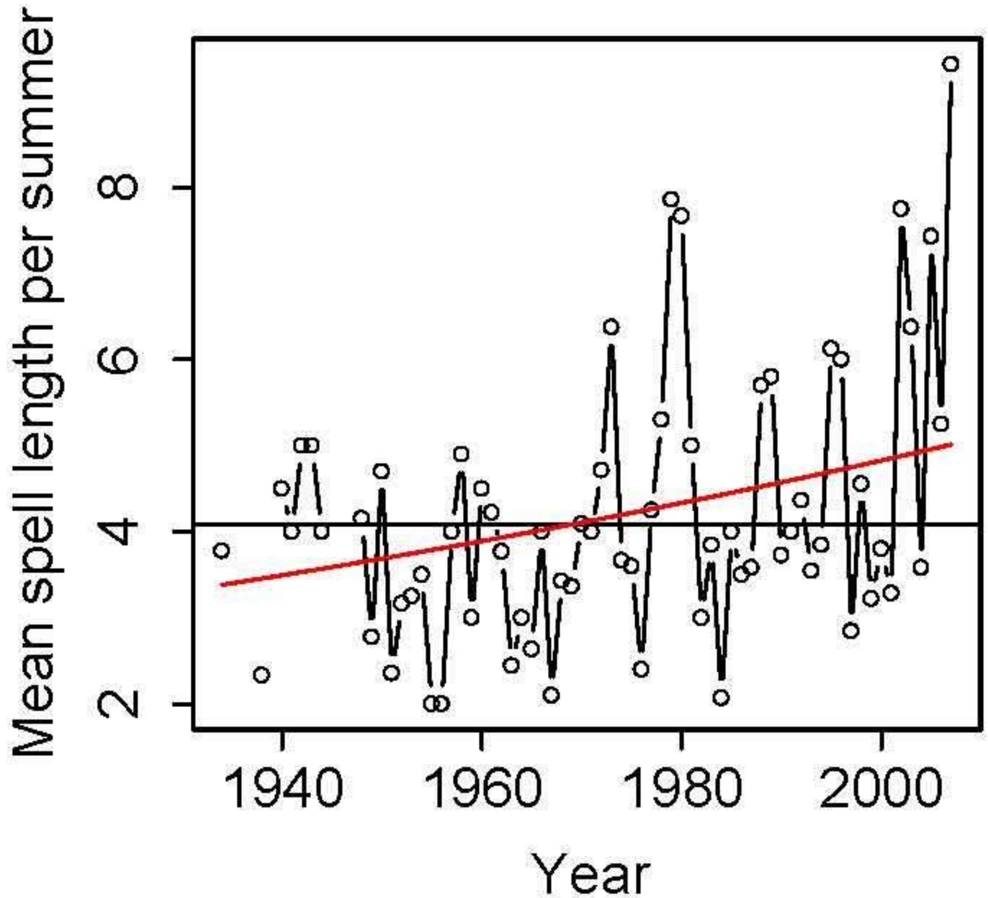
-- Cluster length

Trend in mean of geometric distribution $1/\theta(s)$, year s

-- Cluster maxima (or first excess)

Trend in scale parameter of GP distribution $\sigma^*(s)$, year s

-- Other covariates such as index of atmospheric blocking



Phoenix (GLM with log link, *P*-value ≈ 0.01)

(5) Risk Communication under Stationarity

- **Interpretation of return level $x(p)$ (under stationarity)**

-- Stationarity implies identical distributions

(not necessarily independence)

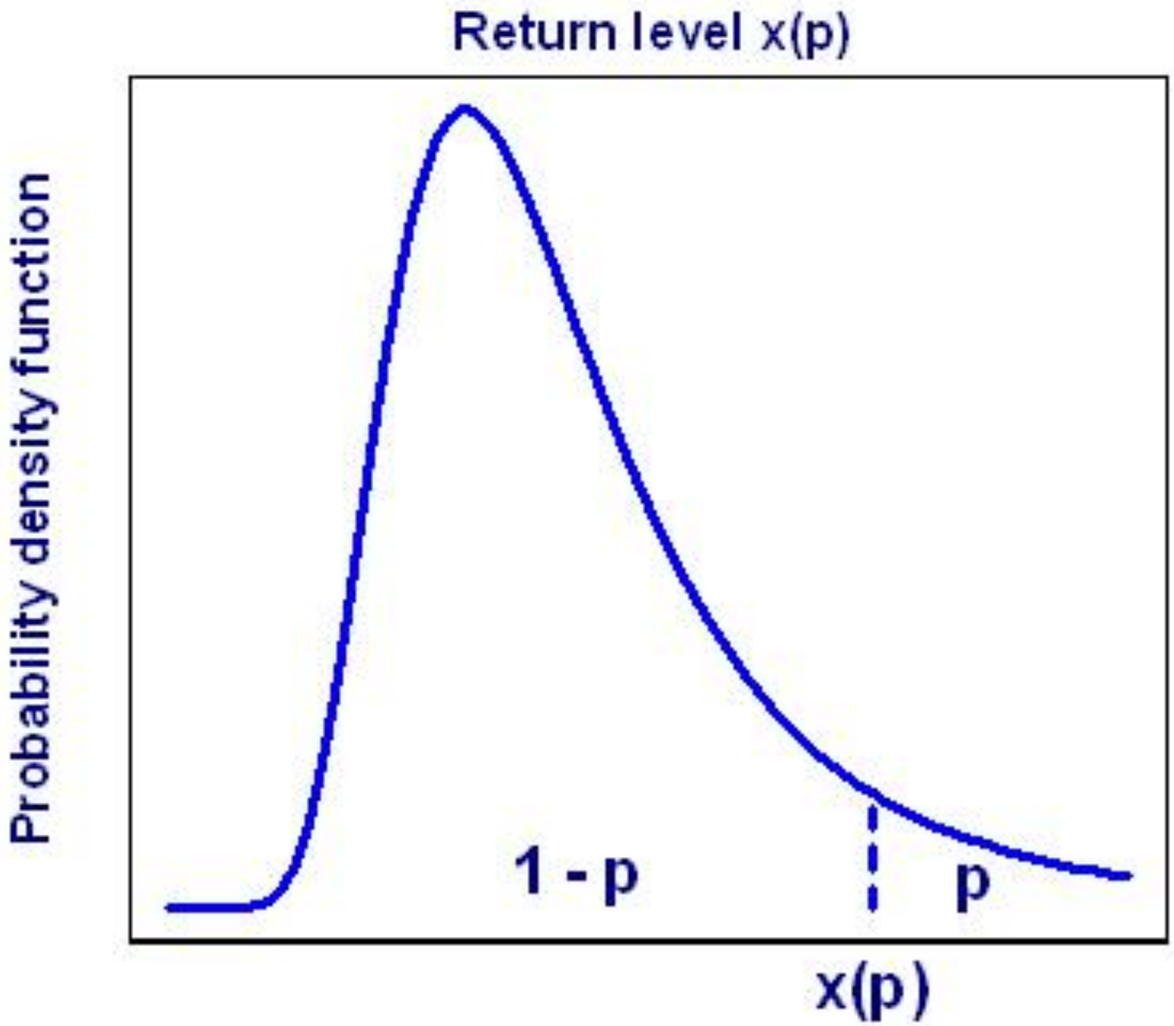
(i) Expected waiting time (under temporal independence)

Waiting time W has geometric distribution:

$$\Pr\{W = k\} = (1 - p)^{k-1} p, \; k = 1, 2, \ldots, \quad E(W) = 1/p$$

(ii) Length of time T_p for which expected number of events = 1

$$1 = \text{Expected no. events} = T_p \, p, \quad \text{so } T_p = 1/p$$



Return level x(p)
Probability density function
1 - p
p
x(p)

(6) Risk Communication under Nonstationarity

- **Options**

-- Retain one of these two interpretations

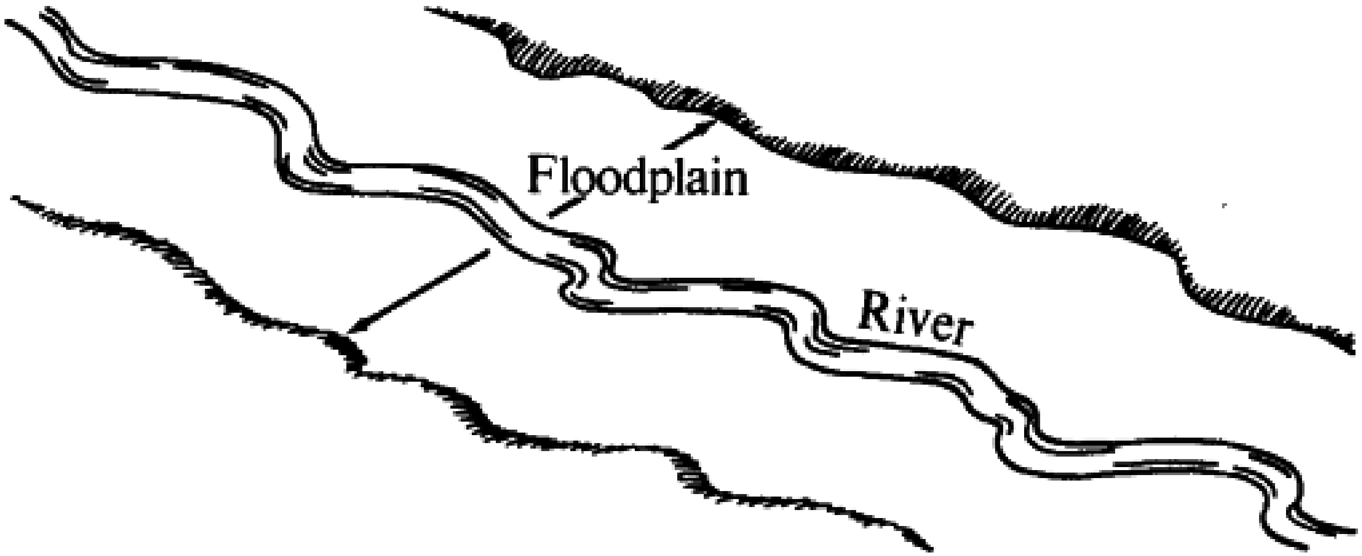
Not clear which one is preferable:

Property (ii) is easier to work with (like average probability)

Property (i) may be more meaningful for risk analysis

-- Switch to "effective" return period and "effective" return level

(i. e., quantiles varying over time)



Moving flood plain from year-to-year (not necessarily feasible?)

- **Alternative concept**

-- Extreme event $X_t > u$

-- Choose threshold *u* to achieve desired value of

Pr{One or more events over time interval of length *T*}

-- Under stationarity (and temporal independence)

As an example, if $p = 0.01$ (i. e., 100-yr return level):

Pr{one or more events over 30 yrs} = $1 - (0.99)^{30} \approx 0.26$

Pr{one or more events over 100 yrs} = $1 - (0.99)^{100} \approx 0.63$